

# A $k$ -sum Decomposition of Strongly Unimodular Matrices

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## Abstract

We present a new decomposition result for strongly unimodular matrices based on the fact that they represent regular matroids over the real field. Specifically, strongly unimodular matrices are shown to be closed under  $k$ -sums for  $k = 1, 2$  implying a decomposition into highly connected blocks which are proved to have a special structure.

## 1 Introduction

Totally unimodular (TU) matrices form an important class of matrices for integer and linear programming due to the integrality properties of the associated polyhedron. A matrix  $A$  is *totally unimodular* if each square submatrix of  $A$  has determinant 0, +1, or  $-1$ . The class of TU matrices has been studied extensively and combinatorial characterizations for these matrices can be found in [7, 9, 10]. An important subclass of TU matrices is defined as follows. A matrix  $A$  is *strongly unimodular* (SU) if: (i)  $A$  is TU, and (ii) every matrix obtained from  $A$  setting a  $\pm 1$  entry to 0 is also TU. Strongly unimodular matrices have appeared several times in the literature [2, 4, 6] since they were first introduced in [3]. Another subclass of TU matrices discussed in this paper is the class of network matrices. A *network matrix* may be viewed as an edge-path matrix of a directed graph with respect to a particular tree of the graph; results regarding network matrices can be found in [7, 9, 10]. Seymour has shown in [11] that network matrices and their transposes are the main building blocks for TU matrices. In this paper we show that SU matrices are closed under  $k$ -sum operations for  $k = 1, 2$  implying a decomposition into smaller SU matrices representing 3-connected regular matroids, for which matrices we provide a characterization.

The rest of the paper is organized as follows. In section 2, we show that SU matrices are closed under the  $k$ -sum operations ( $k = 1, 2$ ) and thereby, they can be decomposed into smaller SU matrices via these operations. The special structure of these smaller matrices is discussed in section 3. Finally, we assume that the reader is aware of basic notions of graph theory and matroid theory. Our references for graph theory are [1, 5] and for matroid theory are [8, 13].

## 2 $k$ -sums of strongly unimodular matrices

The following two results (Lemmata 2.1 and 2.2) can be obtained easily from the definition of SU matrices and the fact that TU matrices are closed under deletions of rows and columns. The proof of Lemma 2.1 is straightforward and is omitted.

**Lemma 2.1.** *Every submatrix of a strongly unimodular matrix is strongly unimodular.*

**Lemma 2.2.** *A  $\{0, \pm 1\}$  TU matrix having at most two nonzeros in every column (row) is SU.*

*Proof:* Let  $A$  be a TU matrix with at most two nonzeros in every column. The case in which  $A$  has two nonzeros in every row can be handled in much the same way. Let us change a nonzero of column  $i$  of  $A$  and call  $A'$  the matrix so-obtained. Now every submatrix of  $A'$  either is equal to the corresponding submatrix of  $A$ ; or we can expand the determinant of the submatrix of  $A'$  along column  $i$  (which has one nonzero being  $\pm 1$ ) and observe that the determinant of  $A'$  is actually equal, up to  $\pm 1$  scaling, to the determinant of a submatrix of  $A$ . Thus, in all cases the determinant of any submatrix of  $A'$  is  $\{0, \pm 1\}$  and therefore  $A$  is SU.  $\square$

As shown in the following result, strongly unimodular matrices are closed under some matrix operations.

**Lemma 2.3.** *Strongly unimodular matrices are closed under the following operations:*

- (i) *transposing,*
- (ii) *adding a zero row or column,*
- (iii) *adding a unit column or a unit row, and*
- (iv) *repeating a column or a row*

*Proof:* Part (i) is trivial since the determinant of any submatrix remains unchanged under transposing. For (ii), the addition of a zero column (row) to a matrix  $A$  results to a matrix  $A'$  which is TU, since TU matrices are closed under the addition of a zero column (row). Furthermore, the replacement of any nonzero of  $A'$  by a zero has to take place to the submatrix  $A$  of it. But  $A$  is SU and therefore we have that the matrix so-obtained is a TU matrix plus a zero column (row). The result now follows from the fact that TU matrices are closed under the addition of a zero row (column).

For (iii), let's add a unit column  $a$  to an SU matrix  $A$  and let's call  $A' = [A \ a]$  the matrix so-obtained. The case in which a unit row is added can be handled similarly. If we change the nonzero of column  $a$  to zero then this is equivalent of adding a zero row to a TU matrix and therefore the matrix so-obtained remains TU. If we change any other nonzero of  $A'$  to zero then this has to be an element of the part  $A$  of  $A'$ ; we change such a nonzero to zero and we call  $A'' = [B \ a]$  the new matrix. We shall show that any submatrix of  $A''$  is TU. Obviously, any submatrix of  $B$  is TU because  $A$  is an SU matrix. In the remaining case, we can expand the determinant of a submatrix along column  $a$  and observe that this determinant is a  $\pm 1$  multiple of the determinant of a submatrix of  $B$ .

For (iv), let  $A' = [A \ a_1]$  be an SU matrix and let  $a_1$  be a column of  $A'$  which we repeat in order to construct the matrix  $A_r = [A \ a_1 \ a_1]$ . We note here that the case of repeating a row can be handled in the same way. The only case which has to be examined is the one in which a nonzero element of a column  $a_1$  becomes zero, since for all the other cases all the submatrices of the matrix obtained are easily checked to be TU. Let  $a'_1$  be the matrix obtained from turning a nonzero of a column  $a_1$  to zero, then the only submatrices of  $A'' = [A \ a'_1 \ a_1]$  which has to be examined of being TU are those containing parts of column  $a'_1$  and  $a_1$ , since all the other submatrices are trivially TU. Expanding now the determinant of such a submatrix of  $A''$  along the column  $a'_1$  and also expand the determinant of the same submatrix of  $[A \ a_1 \ a_1]$  along  $a_1$  we see that these two determinants differ by a determinant of a TU matrix. Thus, these determinants differ by 0 or  $\pm 1$ . But the determinant of the submatrix of  $[A \ a_1 \ a_1]$  is equal to zero and therefore we have that the determinant of the corresponding submatrix of  $A''$  is either 0 or  $\pm 1$ .  $\square$

The  $k$ -sum operations ( $k = 1, 2, 3$ ) on matrices are of central importance for our work and are defined as follows:

**Definition 1.** If  $A, B$  are matrices,  $a, d$  are column vectors and  $b, c$  are row vectors of appropriate size in  $\mathbb{R}$  then

$$\textbf{1-sum: } A \oplus_1 B := \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

$$\textbf{2-sum: } \begin{bmatrix} A & a \end{bmatrix} \oplus_2 \begin{bmatrix} b \\ B \end{bmatrix} := \begin{bmatrix} A & ab \\ 0 & B \end{bmatrix}$$

$$\textbf{3-sum: } \begin{bmatrix} A & a & a \\ c & 0 & 1 \end{bmatrix} \oplus_3 \begin{bmatrix} 1 & 0 & b \\ d & d & B \end{bmatrix} := \begin{bmatrix} A & ab \\ dc & B \end{bmatrix} \text{ or } \begin{bmatrix} A & 0 \\ b & 1 \\ c & 1 \end{bmatrix} \oplus^3 \begin{bmatrix} 1 & 1 & 0 \\ a & d & B \end{bmatrix} := \begin{bmatrix} A & 0 \\ D & B \end{bmatrix}$$

where, in the  $\oplus^3$ ,  $b$  and  $c$  are  $\mathbb{R}$ -independent row vectors and  $a$  and  $d$  are  $\mathbb{R}$ -independent column vectors such that  $\begin{bmatrix} b \\ c \end{bmatrix} = [D_1 | \bar{D}]$ ,  $[a | d] = [\frac{\bar{D}}{D_2}]$  and  $\bar{D}$  is a square non-singular matrix. Then,  $D = [a | d] \bar{D}^{-1} \begin{bmatrix} b \\ c \end{bmatrix}$ .

We show that SU matrices are closed under the 1-sum and 2-sum operations.

**Lemma 2.4.** If  $A$  and  $B$  are SU matrices then the matrix  $N = A \oplus_1 B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$  is an SU matrix.

*Proof:* Since  $A$  and  $B$  are TU and from the fact that TU matrices are closed under 1-sums we have that  $N$  is TU. It remains to be shown that if we change a nonzero of the submatrix  $A$  (or  $B$ ) of  $N$  to zero then the matrix  $N' = \begin{bmatrix} A' & 0 \\ 0 & B' \end{bmatrix}$  obtained by this change is TU. Since  $A$  and  $B$  are SU we have that  $A'$  and  $B'$  are TU and from the fact that TU matrices are closed under the 1-sum operation we have that  $N'$  is TU as well.  $\square$

**Lemma 2.5.** If  $A = \begin{bmatrix} A' & a \end{bmatrix}$  and  $B = \begin{bmatrix} b \\ B' \end{bmatrix}$  are SU matrices then the matrix  $N = A \oplus_2 B = \begin{bmatrix} A' & ab \\ 0 & B' \end{bmatrix}$  is an SU matrix.

*Proof:* Since TU matrices are closed under 2-sums we have that the matrix  $N$ , which is the 2-sum of the TU matrices  $A$  and  $B$ , is TU. It remains to be shown that changing a nonzero of  $N$  to zero the matrix  $N'$  so-obtained is also TU. We consider the following two cases separately: (i) we replace a nonzero of the submatrix  $A'$  or  $B'$  of  $N$  by zero, and (ii) we replace a nonzero element of the  $ab$  submatrix of  $N$  by zero.

For case (i) we can assume without loss of generality that we change a nonzero element of  $A'$  to zero and let us call  $\bar{N} = \begin{bmatrix} \bar{A}' & ab \\ 0 & B' \end{bmatrix}$  the matrix so-obtained (the case in which a nonzero element of  $B'$  is changed is similar). Therefore, matrix  $\bar{N}$  is the 2-sum of the matrix  $\begin{bmatrix} \bar{A}' & a \end{bmatrix}$  and  $\begin{bmatrix} b \\ B' \end{bmatrix}$ , where  $\begin{bmatrix} \bar{A}' & a \end{bmatrix}$  is a TU matrix since it is obtained from the SU matrix  $A$  by replacement of a nonzero by a zero, and

$\begin{bmatrix} b \\ B' \end{bmatrix}$  is TU since it is equal to matrix  $B$ . From the fact that TU matrices are closed under 2-sums the result follows.

For case (ii), let  $N'$  be the matrix obtained from changing a nonzero of the  $ab$  part of  $N$  to zero. We shall show that  $N'$  is TU. Initially, we can see that  $N' = \left[ \begin{array}{c|c} A' & a_1 \\ \hline \mathbf{0} & b_1 \end{array} \middle| \begin{array}{c} ab_2 \\ B_1 \end{array} \right]$ , where  $a_1$  contains the nonzero having changed and thus differs from column  $a$  only to that element; furthermore, we assume that  $b_1$  is the first column of  $B'$  and  $B_1$  is the rest of it, i.e.  $B' = [b_1 \ B_1]$ , and that  $B$  is the matrix having as first row the vector  $[1 \ b_2]$ , where the first element of this vector has to be 1 otherwise we could not find a nonzero to change in order to create  $a_1$ , i.e.  $B = \left[ \begin{array}{cc} 1 & b_2 \\ b_1 & B_1 \end{array} \right]$ .

We can easily see that  $N'$  is the 3-sum of the following two matrices

$$\hat{A} = \left[ \begin{array}{cc|cc} A' & a_1 & a & a \\ \hline \mathbf{0} & 1 & 0 & 1 \end{array} \right]$$

$$\hat{B} = \left[ \begin{array}{ccc} 0 & 1 & b_2 \\ b_1 & b_1 & B_1 \end{array} \right]$$

Since TU matrices closed under 3-sums it suffices to show that each of  $\hat{A}$  and  $\hat{B}$  is TU. We know that  $[A \ a \ a]$  is SU because of Lemma 2.3 (iv); moreover, from (iii) of the same Lemma we have that  $\left[ \begin{array}{c|cc} A' & a & a \\ \hline \mathbf{0} & 0 & 1 \end{array} \right]$  is SU. Finally, applying again (iv) of Lemma 2.3 we have that  $A_m = \left[ \begin{array}{c|cc} A' & a & a \\ \hline \mathbf{0} & 1 & 0 \end{array} \right]$  is SU. Thus, changing a specific nonzero from a column  $\left[ \begin{array}{c} a \\ 1 \end{array} \right]$  of  $A_m$  to zero we obtain  $\hat{A}$  which has to be TU.

We know that  $B$  is SU, so by Lemma 2.3, we have that the matrix  $B_m = \left[ \begin{array}{ccc} 1 & 1 & b_2 \\ b_1 & b_1 & B_1 \end{array} \right]$  is SU. Thus, replacing a 1 of a column  $\left[ \begin{array}{c} 1 \\ b \end{array} \right]$  of  $B_m$  we obtain matrix  $\hat{B}$  which has to be TU. Since both  $\hat{A}$  and  $\hat{B}$  are TU the result follows.  $\square$

In what follows we shall make use of the following regular matroid decomposition theorem by Seymour [11].

**Theorem 2.6.** *Every regular matroid  $M$  may be constructed by means of 1-, 2-, and 3-sums starting with matroids each isomorphic to a minor of  $M$  and each either graphic or cographic or isomorphic to  $R_{10}$ .*

The  $R_{10}$  regular matroid is a ten-element matroid, which can be found in [8, 13], and it has two unique totally unimodular compact representation matrices  $B_1$  and  $B_2$ , up to row and column permutations and scaling of rows and columns by  $-1$ .

$$\begin{aligned}
B_1 = & \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 1 & -1 \\ -1 & 1 & 0 & 0 & 1 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \end{bmatrix} \end{matrix} \quad B_2 = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix} \end{matrix} \quad (1)
\end{aligned}$$

A consequence of theorem Theorem 2.6 is the construction Theorem 2.7 for totally unimodular matrices which appears in [12, 13].

**Theorem 2.7.** *Any TU matrix is up to row and column permutations and scaling by  $\pm 1$  factors a network matrix, the transpose of a network matrix, the matrix  $B_1$  or  $B_2$  of (1), or may be constructed recursively by these matrices using matrix 1-, 2- and 3-sums.*

According to Theorem 2.7, the building blocks for totally unimodular matrices are the network matrices and their transposes as well as matrices  $B_1$  and  $B_2$  of (1).

**Lemma 2.8.**  *$B_1$  and  $B_2$  are not SU.*

*Proof:* If we make the value of the  $(4, 3)^{\text{th}}$ -element of  $B_1$  from  $-1$  to  $0$  then in the matrix so-obtained the  $3 \times 3$  submatrix defined by rows 3, 4 and 5 and columns 2, 3 and 4 has determinant equal to  $+2$ . Therefore,  $B_1$  is not SU. Similarly, if we make the value of the  $(4, 1)^{\text{th}}$ -element of  $B_2$  from  $+1$  to  $0$  then in the matrix so-obtained, the  $3 \times 3$  submatrix defined by rows 3, 4 and 5 and columns 1, 4 and 5 has determinant equal to  $-2$  and thus,  $B_2$  is not SU.  $\square$

By Theorem 2.7 and Lemma 2.8 we obtain the following result.

**Theorem 2.9.** *Any SU matrix is up to row and column permutations and scaling by  $\pm 1$  factors a network matrix, the transpose of a network matrix, or may be constructed recursively by these matrices using matrix 1-, 2- and 3-sums.*

The following theorem, known as the splitter theorem for regular matroids, is one of the most important steps which led to the regular matroid decomposition theorem [11].

**Theorem 2.10.** *Every regular matroid can be obtained from copies of  $R_{10}$  and from 3-connected minors without  $R_{10}$  minors by a sequence of 1-sums and 2-sums.*

Combining Lemmata 2.4, 2.5 and 2.8 and Theorem 2.10 we can now state the main result of this section.

**Theorem 2.11.** *A matrix is strongly unimodular if and only if it is decomposed via 1- and 2-sums into strongly unimodular matrices representing 3-connected regular matroids without  $R_{10}$  minors.*

In view of Theorem 2.11 we can see that an SU matrix can be decomposed via 1-sums and 2-sums into a special class of SU matrices. This class will be characterized in the following section.

### 3 SU matrices of 3-connected regular matroids

By Theorem 2.11 we have that SU matrices are decomposed into smaller SU matrices which represent 3-connected regular matroids without  $R_{10}$  minors. In this section we shall characterize the structure of these smaller matrices in Theorem 3.4.

It is known that any 3-connected binary matroid contains the wheel matroid  $\mathcal{W}_3$  as a minor (Lemma 5.2.10 in [13]). In the following result we show that there exist two TU representation matrices for  $\mathcal{W}_3$ , one SU and one non-SU.

**Lemma 3.1.** *Up to row and column permutations and scaling by  $-1$ , the matroid  $\mathcal{W}_3$  has two different totally unimodular compact representation matrices, namely*

$$1. \text{ an SU representation } N_1 = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \text{ and}$$

$$2. \text{ a non-SU representation } N_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

*Proof:* Since the graphic matroids are uniquely representable over any field, given a TU compact representation of  $\mathcal{W}_3$  we can obtain any other compact representation by row and column permutations, scaling of rows and columns by  $-1$  and pivoting. Since  $\mathcal{W}_3$  is a graphic matroid, each of its TU compact representation matrices is a network matrix as well. Pivoting in a network matrix results to a network matrix with respect to another tree of the same graph. Specifically, up to graph isomorphism, graph  $\mathcal{W}_3$  has two different trees which are depicted in Figure 3, where the bold edges correspond to the tree edges. Thus, up to row and column permutations and scaling by  $-1$ , there are two different network

matrices representing  $\mathcal{W}_3$ ; namely:  $N_1 = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ , and  $N_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ . It is now easy to see

that if we replace any nonzero of  $\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$  by a 0 then all the matrices so-obtained are TU. On

the other hand, if we replace the nonzero at third row and second column of  $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$  by 0 then

the matrix so-obtained is not TU.

□

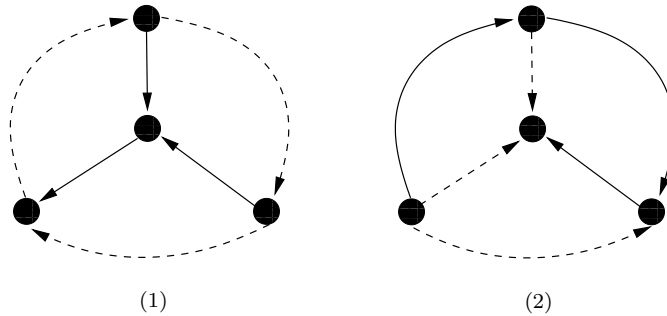


Figure 1: The two possible network representations of  $\mathcal{W}_3$ , where (1) gives rise to an SU network matrix while (2) gives rise to a non-SU network matrix.

We shall now prove the following important theorem which shows that SU representation matrices of 3-connected regular matroids can not have certain  $2 \times 2$  matrices as submatrices.

**Theorem 3.2.** *If  $N$  is an  $m \times n$  representation matrix ( $m, n \geq 3$ ) of a 3-connected regular matroid containing, up to row and column permutations and scalings by  $-1$ , the submatrix  $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ , then  $N$  is not SU.*

*Proof:* Since  $N$  is the representation matrix of a connected matroid we have that it has an  $M(W_2)$  minor (see Lemma 5.2.10 in [13]). Furthermore, the matrix  $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$  under any row and column permutations and scalings by  $-1$  factors displays  $M(W_2)$ . Enlarge this  $2 \times 2$  submatrix to a maximal submatrix containing only 1s. Let us call  $D$  that submatrix and index its rows and columns by  $R$  and  $S$ , respectively. Furthermore, in the partitioned  $N$  of (2) each row of the submatrix  $U$  and each column of the submatrix  $V$  is assumed to be nonzero. From our assumption that  $D$  is maximal we have that each row and each column of  $U$  and  $V$ , respectively, must have at least one zero element.

$$N = \begin{array}{c} R \\ P \end{array} \begin{array}{c} S \quad Q \end{array} \begin{array}{|c|c|c|} \hline D & V & 0 \\ \hline U & 0/\pm 1 & 0/\pm 1 \\ \hline 0 & 0/\pm 1 & 0/\pm 1 \\ \hline \end{array} \quad (2)$$

Let  $BG(N)$  be the bipartite graph of  $N$  and let  $F$  be its subgraph obtained from the deletion of the edges corresponding to the 1s of  $D$ . By the proof of Lemma 5.2.10 and since  $N$  is the representation matrix of a 3-connected regular matroid, we have that there must exist a path in  $F$  connecting a vertex of  $R$  with a vertex of  $S$  which, due to the bipartiteness of  $F$ , has to be of odd length. If we assume that the length of that path is 3 then the matrix  $N_2$  of Lemma 3.1 is a submatrix of  $N$ , which implies that  $N$  is not SU.

If the shortest path connecting a vertex of  $R$  with a vertex of  $S$  has length greater than 3 then we will show that the matrix  $N$  is also non-SU. Let's say that the shortest path lies between the vertices  $r_2$  and  $s_2$  of  $R$  and  $S$ , respectively (see Figure 2). Then  $N$  will have the following submatrix  $M$ :

$$M = \begin{array}{c} r_2 \\ p_n \\ p_{n-1} \\ \vdots \\ p_1 \\ r_1 \end{array} \begin{array}{c} q_1 \quad q_2 \quad \dots \quad q_n \quad s_2 \quad s_1 \end{array} \begin{array}{|c|c|c|c|c|c|c|c|} \hline \pm 1 & 0 & 0 & \dots & 0 & 0 & \pm 1 & \pm 1 \\ \hline \pm 1 & \pm 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ \hline 0 & \pm 1 & \pm 1 & \dots & 0 & 0 & 0 & 0 \\ \hline \dots & & \dots & & \dots & & \dots & \\ \hline 0 & 0 & 0 & \dots & 0 & \pm 1 & \pm 1 & 0 \\ \hline 0 & 0 & 0 & \dots & 0 & 0 & \pm 1 & \pm 1 \\ \hline \end{array}$$

where  $\{r_1, r_2\} \in R$ ,  $\{s_1, s_2\} \in S$ ,  $\{p_1, \dots, p_n\} \in P$  and  $\{q_1, \dots, q_n\} \in Q$ . Moreover, we have that  $M$  will have no zeros in the main diagonal and in the diagonal below the main because of the path existing between  $r_2$  and  $s_2$  (see Figure 2). The submatrix of  $M$  having rows indexed by  $r_1$  and  $r_2$  and columns indexed by  $s_1$  and  $s_2$  is full of ones because it is submatrix of  $D$ . Furthermore, we have zeros in the position indexed by  $r_1$  and  $q_1$  and in the position indexed by  $p_1$  and  $s_1$  because we can assume that there exists at least one vertex of  $R$  not being adjacent to  $q_1$ , which we call  $r_1$ , and similarly we can assume that there exists a vertex of  $S$  not being adjacent to  $p_1$ , which we call  $s_1$ . All the other zeros

in  $M$  are due to the fact that the path between  $r_2$  and  $s_2$  is the shortest between a vertex of  $R$  and a vertex of  $S$  in the graph  $F$ .

We shall now show that matrix  $M$  is not SU. If we expand the determinant of  $M$  along the first row then this determinant is equal to the sum of the determinants of three TU matrices being triangular with no zero in the diagonal. Therefore, it is easy now to see that there exists a nonzero in the first row of  $M$  such that if we replace it by a zero and expand the determinant of the matrix so-obtained along the first row then we have that the determinant of this matrix will be 2 or  $-2$ . Therefore,  $N$  has a submatrix  $M$  being non-SU and by Lemma 2.1,  $N$  is not SU.  $\square$

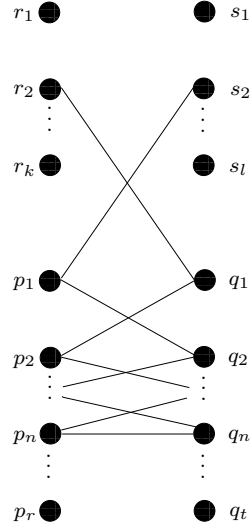


Figure 2: The shortest path from a vertex of  $R$  to a vertex of  $S$  in the graph  $BG(N)$ ; note that all the non-path edges are not depicted.

Crama et al. in [4] proved that if  $A$  is an SU matrix then we can partition its rows as stated in the following theorem.

**Theorem 3.3.** *If  $A$  is an SU matrix, then there exists a partition  $(S_1, \dots, S_k)$  of the rows of  $A$  with the following properties:*

- (i) *every column of  $A$  has 0, 1 or 2 nonzero entries in each  $S_i$ , for  $i = 1, \dots, k$ ;*
- (ii) *if a column has exactly one nonzero entry in some  $S_i$ , then all its entries in  $S_{i+1}, \dots, S_k$  are zeros.*

Since by Lemma 2.3(i), SU matrices are closed under taking the transpose we can restate Theorem 3.3 for the columns of an SU matrix. Consider an SU matrix  $A'$  and let  $\mathcal{S} = (S_1, S_2, \dots, S_k)$  be the partition of its rows as determined by Theorem 3.3 and  $\mathcal{T} = (T_1, T_2, \dots, T_l)$  be the partition of the rows of the transpose of  $A'$  as determined by Theorem 3.3. Then by permuting rows and columns of  $A'$  we can obtain the following SU matrix  $A$ :



$$A = \begin{array}{c|cccc} & T_1 & T_2 & \cdots & T_l \\ \hline S_1 & A_{1,1} & A_{1,2} & \cdots & A_{1,l} \\ S_2 & A_{2,1} & A_{2,2} & \cdots & A_{2,l} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ S_k & A_{k,1} & A_{k,2} & \cdots & A_{k,l} \end{array} \quad (3)$$

where we have that each  $A_{i,j}$  is the submatrix of  $A'$  defined by the rows of  $S_i$  and columns of  $T_j$ .

**Theorem 3.4.** *Let  $A$  be an  $SU$  matrix representation of a 3-connected regular matroid being in the form of (3). Then the following hold:*

- (i)  $A_{1,1}$  has 0 or 2 nonzeros in each column and row
- (ii) each column of  $A_{1,j}$  has 0 or 2 nonzeros and each row of  $A_{i,1}$  has 0 or 2 nonzero elements
- (iii) if an  $A_{i,j}$  has 2 nonzeros in each column and each row then, up to row and column permutations,

$$A_{i,j} = \begin{bmatrix} \pm 1 & & & & \pm 1 \\ \pm 1 & \pm 1 & & & \\ & \pm 1 & \vdots & & \\ & & \vdots & & \\ & & & \pm 1 & \\ & & & \pm 1 & \pm 1 \end{bmatrix}$$

*Proof:* For (i), it is enough to observe that if there was a column (row) of  $A_{i,j}$  with exactly one nonzero, then by Theorem 3.3 this column (row) would be a unit column (row). This would mean that the matroid represented by  $A$  has a 2-separation. This is in contradiction with our hypothesis that this matroid is 3-connected.

For (ii), if a column (row) of some  $A_{1,j}$  ( $A_{i,1}$ ) had exactly one nonzero then by Theorem 3.3 it would be a unitary column (row). Again this is in contradiction with our hypothesis that the matroid represented by  $A$  is 3-connected.

For (iii), from Theorem 3.2 we have that  $A_{i,j}$  has can not the matrix  $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$  as submatrix. It is now straightforward to see that  $A_{i,j}$  has the form described in (iii).  $\square$

## References

- [1] J.A. Bondy and U.S.R. Murty. *Graph Theory*. Springer, 2007
- [2] M. Conforti and M.R. Rao. Structural properties and recognition of restricted and strongly unimodular matrices. *Mathematical Programming*, 38:17–27, 1987.
- [3] Y. Crama, P.L. Hammer and T. Ibaraki. Strong unimodularity for matrices and hypergraphs. *Discrete Applied Mathematics*, 15:221–239, 1986.
- [4] Y. Crama, M. Loeb and S. Poljac. A decomposition of strongly unimodular matrices into incidence matrices of digraphs. *Discrete Mathematics*, 102:143–147, 1992.

- [5] R. Diestel. *Graph Theory*. Springer, 2005.
- [6] M. Loebl and S. Poljac. A hierarchy of totally unimodular matrices. *Discrete Mathematics*, 76:241–246, 1989.
- [7] G.L. Nemhauser and L.A. Wolsey. *Integer and Combinatorial Optimization*. Wiley, 1988.
- [8] J. Oxley. *Matroid Theory*. Oxford University Press, 2006.
- [9] A. Schrijver. *Theory of Linear and Integer Programming*. Wiley, 1986.
- [10] A. Schrijver. *Combinatorial Optimization: Polyhedra and Efficiency*. Springer, 2004.
- [11] P.D. Seymour. Decomposition of regular matroids. *Journal of Combinatorial Theory Series B*, 28:305–359, 1980.
- [12] P.D. Seymour. Matroid Minors. In *Handbook of Combinatorics, Volume I*, 527–550, Elsevier, 1980.
- [13] K. Truemper. *Matroid Decomposition*. Leibniz, 1998.